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Surf-skimmer planing hydrodynamics

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Matched asymptotic expansions are used to analyse the flow past a two-dimensional planing surface in shallow water. A simple momentum conservation relation is obtained connecting leading-edge height, trailing-edge height, and ambient water depth, from which the (initially unknown) wetted length can be determined. This relationship is confirmed by an explicit solution for the flow in the splash zone near the leading edge. The theory is used to discuss the dynamics of a freely skimming board carrying a given weight whose point of application is a given distance ahead of the trailing edge.

1. Introduction

A surf skimmer (or 'skimboard') is a flat circular disk about a metre in diameter, which seems to be able to travel for quite long distances (5–10 m) in very shallow (1–2 cm) water, carrying the weight of a human at speeds like 2–4 m/s. Clearly the hydrodynamics of the flow of water beneath the skimmer must somehow be such as to produce high lift at low drag.

In the present paper, as in the only apparent previous study, by Edge (1968), we assume one-dimensional flow and neglect gravity. The latter assumption simply means that hydrostatic forces are negligible compared to hydrodynamic forces, which is justified by the fact that the Froude number takes quite high values, typically 5 to 10, according to the velocity and depth scales mentioned above. However, there are nevertheless some finite Froude-number questions, to which we address ourselves below.

In order to justify the one-dimensional flow assumption, we must first neglect all lateral flow components, in spite of the fact that the board has a circular plan form. Observation of actual skimmers in action suggests that streamlines are diverted laterally not more than about 30°, and it is reasonable to model at least the most important portion of the flow near the centreplane as if it were confined to that plane. The fact that lateral flow actually occurs is however of vital importance, as we shall see, even though we make no attempt here to compute it. If it is desired to incorporate lateral-flow considerations into the solution, the present work has a relatively easy extension to a 'stripwise' theory analogous to aerodynamic lifting-line theory, and a harder extension to a fully three-dimensional theory analogous to lifting-surface theory (see Tuck 1983; Read 1989).

Having neglected lateral flow, the resulting two-dimensional flow then becomes approximately one-dimensional because of the small depth-to-chord ratio, so that the vertical velocity component is neglected relative to the streamwise component u, in a frame of reference fixed in the skimmer. One-dimensional continuity implies that uh is constant, where h is the local water depth beneath the skimmer. The

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appropriate constant can be written as $u_T h_T$, if u_T and h_T are the values of velocity and depth at the trailing edge.

Then (neglecting viscosity) Bernoulli's equation gives the pressure as

$$p = \text{constant} - \tfrac{1}{2}\rho u^2 = p_0 + \tfrac{1}{2}\rho u_T^2 \left[1 - \frac{h_T^2}{h^2}\right], \tag{1.1}$$

where p_0 is the pressure at the trailing edge, which we can take to be atmospheric pressure. If we assume that u_T and the wetted area A of the skimmer are both known, we can integrate the above pressure to provide the net lift per unit span, and this was done by Edge (1968), after an additional (unnecessary) small-incidence approximation.

It is typical of this kind of extreme ground effect (Tuck 1981) that the resulting lift coefficient is of order one. This simply means that the quantity in square brackets in (1.1) takes order-one values, so that the net lift is of the order of $\rho u_T^2 A$. If at the same time we assume that u_T takes values of the same order of magnitude as the skimmer's speed, we obtain lift values of the order of 100–200 kg-weight, so that human riders are amply supported. Meanwhile there is a small skin-friction drag, plus a splash drag that is minimized by keeping the angle of attack as low as possible. This will be discussed in more detail later.

There are a few unanswered questions if we wish to obtain quantitative values for the forces on the skimmer. The pressure is expressed in terms of u_T, and we still have to relate u_T to the actual speed u_0 of the board, i.e. to the apparent free-stream speed at upstream infinity. Edge (1968) states that, 'for the board to remain afloat, the water entering at the front must come out at the back'. The implication of this statement is that u_T is determined by mass conservation, i.e.

$$u_T = u_0 h_0 / h_T, \tag{1.2}$$

where h_0 is the undisturbed water depth. Thus, if as we assume, $h_T < h_0$, then $u_T > u_0$, and the water 'leaves with a higher velocity as it is squirted out behind'.

Unfortunately, the assumption $u_T > u_0$ is untenable in the absence of viscous and gravitational effects. If the stream emerging from the trailing edge is of magnitude greater than u_0, there must eventually occur somewhere downstream a hydraulic jump at which the velocity returns to the value u_0 and the height to the value h_0. But as is well known (see e.g. Stoker 1957, p. 326), such a hydraulic jump can only occur when one stream is subcritical and the other is supercritical. If $u_T > u_0$, this means that the basic flow must be subcritical, whereas the present flow is so far into the supercritical range that gravity is negligible. Such a hydraulic jump cannot (and does not) occur.

What actually happens is that the skimmer skims. That is, it removes a portion of the undisturbed layer of water. In a model lacking lateral flow, this presents an apparent paradox, since we must query where the skimmed fluid goes, but in practice it is thrown sideways. In any case, we must abandon the mass conservation relation (1.2) between the trailing-edge velocity and the free-stream speed. At the same time, if we demand that the fluid detaches smoothly from the trailing edge, so that the one-dimensional assumption holds in its neighbourhood, then the only possible flow there is a uniform stream of magnitude u_T. Bernoulli's equation then demands that this stream has the same speed $u_T = u_0$ as the incident flow upstream (relative to the skimmer), i.e. in a fixed frame of reference, the skimmer just leaves the water at rest behind it.

The other major difficulty with the above discussion is that in practice the wetted area A is also not known in advance. That is, even if the trailing edge is sharp, so that its position is known in advance, the leading edge can lie anywhere forward of that point. Indeed, this is the secret of successful skimboard riding. The rider adjusts the position of his centre of gravity on the board so as to control the angle of attack and hence the wetted length. Too much angle of attack leads to too small a wetted length, and hence too little lift; too small an angle of attack also gives too little lift, so the good rider needs to pick the correct compromise. Any analysis that fails to incorporate this phenomenon has missed the whole point of the device.

This need to determine the wetted length as part of the solution is common in planing surface theory (see e.g. Squire 1957; Tuck 1989). Indeed, the present problem is nothing more than the shallow-water limit of the problem of planing on water of finite depth, as solved exactly for the special case of a flat plate by Green (1936). However, the shallow-water limit yields simpler and more general results by use of matched asymptotic expansions.

We provide a derivation of this theory in subsequent sections. It is worth quoting now one of the most important results, namely an explicit formula

$$h_L = 2h_0 - h_T + 2(h_0(h_0 - h_T))^{\frac{1}{2}}, \tag{1.3}$$

for the leading-edge water depth h_L in terms of the trailing-edge depth h_T and the undisturbed depth h_0. This equation essentially determines the wetted length, since if h_T is given, the leading edge of the wetted region must adjust itself until the water depth at that point is given by (1.3).

The actual situation is a little more complicated than that, since h_T is not given, but itself must be determined (together with the angle of attack) by the dynamics of the skimmer and rider. We show here how this can be done, by first computing the lift and moment for a fixed skimmer orientation, and then re-plotting these results with the lift and location of its centre of pressure assumed prescribed, and the orientation as output.

2. Planing formulation

If we simply interpret the surf skimmer as a planing surface of chord (wetted length) l_W on water of depth h_0, all that is needed in principle is the limit when

$$\epsilon = h_0/l_W \tag{2.1}$$

tends to zero, of results available by solving the full three-dimensional planing surface problem, e.g. as done by Green (1936) for a flat plate. However, the theory for finite ϵ is very complicated, and the final results are expressed in terms of complex elliptic functions. Carrying out the small ϵ limit on these results is neither simple mathematically nor illuminating physically. Hence we derive the small ϵ limit directly from the boundary-value problem by matched asymptotic expansions, as follows.

If ϵ is small, then the planing surface must also be at a small angle of attack (or else it hits the bottom), and we assume that the angle of attack α and ϵ are comparable in order of magnitude. In that case there is a finite (neither necessarily large nor small) contraction ratio

$$\lambda = h_T/h_L \tag{2.2}$$

between the trailing and leading depths, as indicated in figure 1. Neither of these depths need equal the undisturbed depth h_0 of water, interpreted now as the depth of a uniform stream u_0 far upstream of a fixed skimmer.

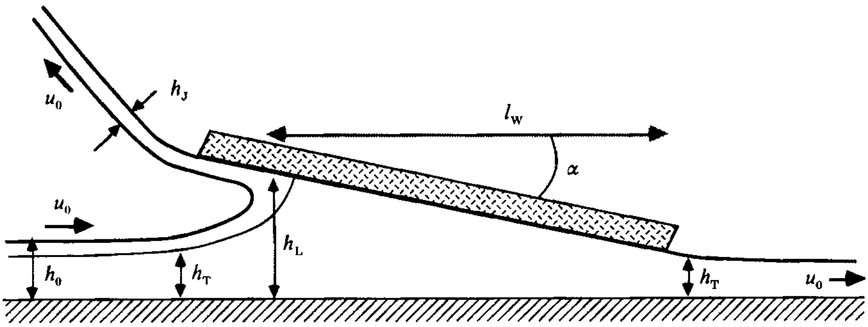

FIGURE 1. Sketch of flow and geometry.

Indeed, our assumption now is the standard one in planing surface theory that the skimmer throws a portion of the incident stream ahead of itself as a jet. This means that there must be a stagnation point near the leading edge of the skimmer, and a submerged stagnation streamline meeting it from upstream. All fluid above the stagnation streamline enters the jet; all below it passes under the skimmer and out at its trailing edge.

If we assume smooth detachment at this trailing edge, then the only possible one-dimensional flow with atmospheric pressure on its free surface at and downstream of the trailing edge is a uniform stream of magnitude u_0 and uniform depth h_T. But then by conservation of mass in the flow beneath the skimmer, the far-upstream height of the stagnation streamline must also be h_T. Hence the ultimate jet thickness must be

$$h_J = h_0 - h_T, \tag{2.3}$$

noting that the constant-speed free-surface condition demands that (in the absence of gravity) the jet ultimately moves as a uniform stream at speed u_0, with a thickness the same as the excess height of the undisturbed free surface over that of the stagnation streamline.

The direction of the ultimate jet is not as clear as is its magnitude. In the absence of gravity and any three-dimensional effects, the jet remains attached to the lower surface of the skimmer so long as that exists. When the jet reaches the forward tip of the skimmer, it becomes a free jet which (in the absence of gravity) continues for ever without further change of direction, although its angle may differ slightly from that of the skimmer, e.g. may involve a small upward deflection. On the other hand, if the skimmer is effectively unlimited in its forward extent, the jet remains attached until gravity finally causes it to fall, but we assume that this occurs so far forward as to have no effect on the main flow. In practice, this would seem to require some form of 'bucket' far upstream to catch the falling jet before it splashed. More realistically, three-dimensional effects would divert the splash away from the most important centreline domain of flow.

Hence whatever jet is created, of thickness h_J, is lost to the flow beneath the skimmer, and in that sense, mass is not conserved. What then do we mean by the leading-edge height h_L? Where indeed is the leading edge, if the skimmer is effectively wetted infinitely far forward by an attached jet? The issue of dynamic determination of the location of the leading edge is of profound importance in all planing surface theory, no less here.

What is clear is that, in the limit as both $\alpha \to 0$ and $\epsilon \to 0$, the zone where the jet and the stagnation point are created has a small longitudinal (horizontal) extent

$O(h_0) \ll l_W$. For example (see Appendix) the point of near-maximum curvature where the jet is being bent back, and the free surface is locally vertical, is a distance $O(h_J) = O(h_0)$ ahead of the stagnation point. This zone is one where horizontal and vertical velocity components are comparable, but it is a zone of vanishing size relative to the horizontal lengthscale l_W of the skimmer. Hence it is an 'inner region' in the sense of matched asymptotic expansions, whose detail we now examine.

3. Inner expansion near the leading edge

The flow in the leading-edge region is as sketched in figure 2. Note that, in the limit as $\alpha \to 0$, the skimmer looks like a flat plate parallel to the bottom, i.e. the flow takes place in a channel of width h_L between parallel plates. It is convenient to consider h_L as the height of the stagnation point; however, again note that the height is uniform throughout the (small) horizontal extent of the leading-edge zone.

The flow that takes place in this inner region is as follows. An incoming stream u_0 of height $h_0 < h_L$ from $x = -\infty$ is partly deflected back to $x = -\infty$, and partly flows onward to $x = +\infty$, where it fills the whole channel but moves at reduced speed u_L. If the jet thickness is $h_J = h_0 - h_T$, conservation of mass indicates that

$$u_0 h_T = u_L h_L. \tag{3.1}$$

The flow in figure 2 could not occur unless some force was applied in the $-x$-direction to deflect the jet. This is manifest in a (positive) difference between the pressure p_L in the outgoing stream at $x = +\infty$ and the (atmospheric) pressure p_0 in the incoming stream from $x = -\infty$. Bernoulli's equation yields this pressure difference as

$$p_L - p_0 = \tfrac{1}{2}\rho(u_0^2 - u_L^2), \tag{3.2}$$

noting that $h_T < h_L$ implies $u_L < u_0$ and $p_L - p_0 > 0$.

Now horizontal momentum balance demands that the net force $(p_L - p_0) h_L$ exerted by the pressure difference balances the net momentum loss due to the thrown-forward jet, namely that of magnitude $\rho u_0^2 h_J$ in the jet itself (of speed u_0 and thickness h_J), plus the difference $\rho u_0^2 h_0 - \rho u_L^2 h_L$ in the incoming and outgoing stream momenta. (This momentum balance can be given a more formal derivation by integrating Euler's equation on a suitable control surface.) That is,

$$\tfrac{1}{2}\rho(u_0^2 - u_L^2) h_L = \rho u_0^2 h_J + \rho u_0^2 h_0 - \rho u_L^2 h_L, \tag{3.3}$$

or, using (2.3) for the jet thickness h_J,

$$\tfrac{1}{2}u_0^2 h_L + \tfrac{1}{2}u_L^2 h_L = 2u_0^2 h_0 - u_0^2 h_T. \tag{3.4}$$

Eliminating h_T by use of (3.1) yields

$$\tfrac{1}{2}h_L(u_0 + u_L)^2 = 2u_0^2 h_0, \tag{3.5}$$

or

$$\frac{u_L}{u_0} = 2\left(\frac{h_0}{h_L}\right)^{\frac{1}{2}} - 1. \tag{3.6}$$

Equation (3.6) is of fundamental importance in this problem. As it stands, it provides the outgoing flow speed relative to that incoming, as an explicit function of the ratio between outgoing and incoming water depths in the leading-edge region.

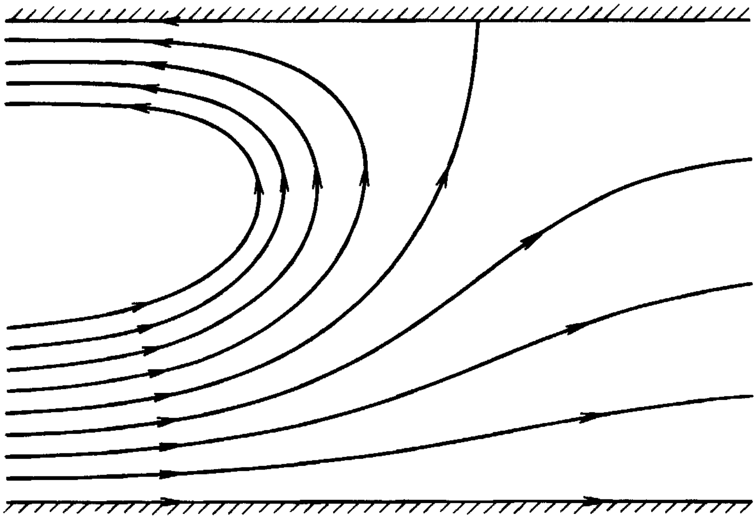

FIGURE 2. Flow in the inner (leading-edge) region. Streamlines shown are exact computations for a contraction ratio of 0.5.

Alternatively, we can re-introduce the depth h_T by (3.1), showing that the same function determines the contraction ratio λ, i.e.

$$\lambda = \frac{h_T}{h_L} = 2\left(\frac{h_0}{h_L}\right)^{\frac{1}{2}} - 1, \tag{3.7}$$

which provides a relation connecting all three depths h_0, h_L, h_T. This relation has already been quoted in §1, in the form of an equation (1.3) for h_L as an explicit function of h_0 and h_T Note the equivalent formulae

$$\frac{h_L}{h_0} = \frac{4}{(1+\lambda)^2}, \tag{3.8}$$

and

$$\frac{h_T}{h_0} = \frac{4\lambda}{(1+\lambda)^2}, \tag{3.9}$$

expressing the leading and trailing depths (relative to the undisturbed depth)as functions of the contraction ratio λ.

An alternative method for establishing the relations (3.6)–(3.9) is to solve the problem sketched in figure 2 explicitly, and this is done in the Appendix.

4. Outer region beneath the skimmer

Now we must match the leading-edge flow whose properties were outlined in the previous section to flow in an outer region of horizontal extent $O(l_W)$, as indicated in figure 3. In this region, u_L is to be interpreted as the apparent entry velocity at $x = x_L$ to a channel of decreasing height $h(x)$, which ends at the trailing edge $x = x_T$, where the flow must have returned to its free-stream velocity u_0. Since the horizontal scale $l_W = x_T - x_L$ is so far in excess of the vertical scale h, the flow in this channel is one-dimensional, and thus its horizontal velocity $u(x)$ satisfies the one-dimensional continuity equation

$$u(x)\,h(x) = \text{constant} = u_0\,h_T. \tag{4.1}$$

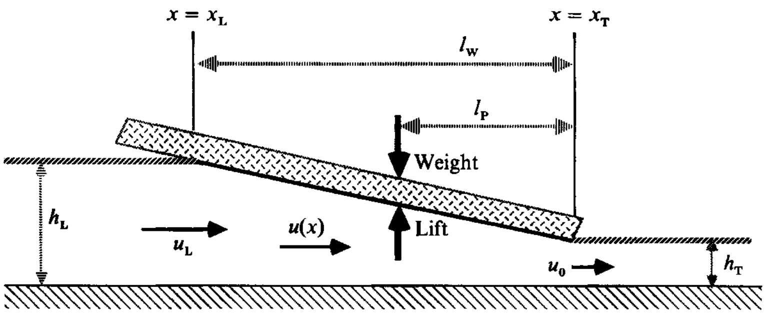

FIGURE 3. Outer flow parameters.

Now the pressure in this one-dimensional flow beneath the skimmer is

$$p(x) = p_0 + \tfrac{1}{2}\rho[u_0^2 - u(x)^2] = p_0 + \tfrac{1}{2}\rho u_0^2\left[1 - \frac{h_T^2}{h(x)^2}\right]. \tag{4.2}$$

Given any clearance function $h(x)$ is determined by the given under-surface shape of the skimmer, and the trailing-edge height $h_T - h(x_T)$, (4.2) provides the pressure everywhere on the skimmer's wetted surface aft of the leading edge, and hence enables determination of the forces on it. The contribution to these forces from the jet is negligible even if it wets the skimmer well forward of x_L. The jet has a pressure different from atmospheric only in the vanishingly-small leading-edge region. Thus the net lift force per unit span is

$$F = \int_{x_L}^{x_T} (p - p_0)\,\mathrm{d}x = \tfrac{1}{2}\rho u_0^2 \int_{x_L}^{x_T} \left[1 - \frac{h_T^2}{h(x)^2}\right] \mathrm{d}x, \tag{4.3}$$

and the anticlockwise pitching moment about the trailing edge is

$$M = \tfrac{1}{2}\rho u_0^2 \int_{x_L}^{x_T} (x - x_T)\left[1 - \frac{h_T^2}{h(x)^2}\right] \mathrm{d}x. \tag{4.4}$$

These formulae provide known forces only if we already know the location $x = x_L$ of the leading edge. But this is defined as the value of x at which the height is h_L, i.e. by

$$h_L = h(x_L). \tag{4.5}$$

That is, if the shape function $h(x)$ is given, and we use (3.7) as a formula to determine h_L, given both h_0 and h_T, we can solve (4.5) for x_L, hence find the wetted length $l_W = x_T - x_L$, and the force and moment. Once again, it is the fact that the wetted length is not known in advance that provides the unique feature of this problem. Since the forces on the skimmer (including viscous drag via skin friction) are particularly sensitive to wetted length, this feature is of vital importance in the skimmer's dynamics, and has not been taken into account in previous analyses such as that of Edge (1968).

In fact, the above is not even the end of the story regarding the true input–output nature of the problem, since in practice even the trailing-edge height h_T is not known in advance. What we do know is the net lift F, balancing the weight, and the net moment M via the centre of pressure (known from the point of application of that weight, i.e. the location of the rider's foot relative to the trailing edge). Nevertheless, let us temporarily assume x_L and x_T known, and determine F and M, inverting the results at the end.

For example, suppose that the skimmer is a rigid flat plate at angle of attack α (itself temporarily assumed known, but really an output quantity). Then

$$h(x) = h_T - \alpha(x - x_T), \tag{4.6}$$

and the integrals (4.3), (4.4) for the force and moment can be evaluated explicitly. The result (Tuck 1981) is

$$F = \tfrac{1}{2}\rho u_0^2 l_W(1-\lambda), \tag{4.7}$$

and

$$M = \tfrac{1}{4}\rho u_0^2 l_W^2(\lambda - 1 - \mu), \tag{4.8}$$

where λ is the contraction ratio, as defined in (2.1), and μ is a specific function of λ, namely

$$\mu = \lambda\left(\frac{1+\lambda}{1-\lambda}\right) + 2\left(\frac{\lambda}{1-\lambda}\right)^2 \log\lambda. \tag{4.9}$$

Thus the centre of pressure lies at $x = x_P$, a distance $l_P = x_T - x_P$ ahead of the trailing edge, where

$$l_P = -M/F = \tfrac{1}{2}l_W\left(1 + \frac{\mu}{1-\lambda}\right). \tag{4.10}$$

It is convenient to now take l_P as our fundamental length, defining a lift coefficient C_F via

$$F = \tfrac{1}{2}\rho u_0^2 l_P C_F. \tag{4.11}$$

If the speed u_0, the weight F, and the centre of pressure ($\equiv$ centre of gravity) location l_P relative to the trailing edge are all given quantities for a fluid of given density ρ, then so is this coefficient C_F, and it is appropriate to express all output quantities in terms of this true input parameter C_F.

But (4.7), (4.10) and (4.11) together imply that

$$C_F = \frac{2(1-\lambda)^2}{1-\lambda+\mu} \tag{4.12}$$

is a known function of λ. Hence, given C_F, we now know the contraction coefficient λ. Then (3.8) and (3.9) give the leading and trailing depths, relative to the undisturbed depth. Finally, the angle of attack is given by

$$\alpha = \frac{h_L - h_T}{l_W},$$

i.e. the scaled angle of attack of $\alpha l_P/h_0$ is also a known function of λ, and hence of C_F.

5. Results

Figures 4 and 5 show output quantities as functions of C_F. These are computed by first determining all variables (including C_F) as functions of λ, then re-plotting as functions of C_F. In figure 4 we give the vertical scale ratios h_L/h_0 and h_T/h_0, and in figure 5 the horizontal scale ratios l_W/l_P and $\alpha l_P/h_0$.

When the skimmer is lightly loaded, i.e. C_F is small, obviously it penetrates the water surface only a small amount at its trailing edge, with a small angle of attack and a small rise of water level at the leading edge. In this small-disturbance limit, the centre of pressure location is at the one-third chord point (Tuck 1981), so that $l_W/l_P = 1.5$. This is favourable for flight, since there is minimum drag and vanishingly small splash.

As C_F increases, the angle of attack increases rapidly, as does the leading-edge

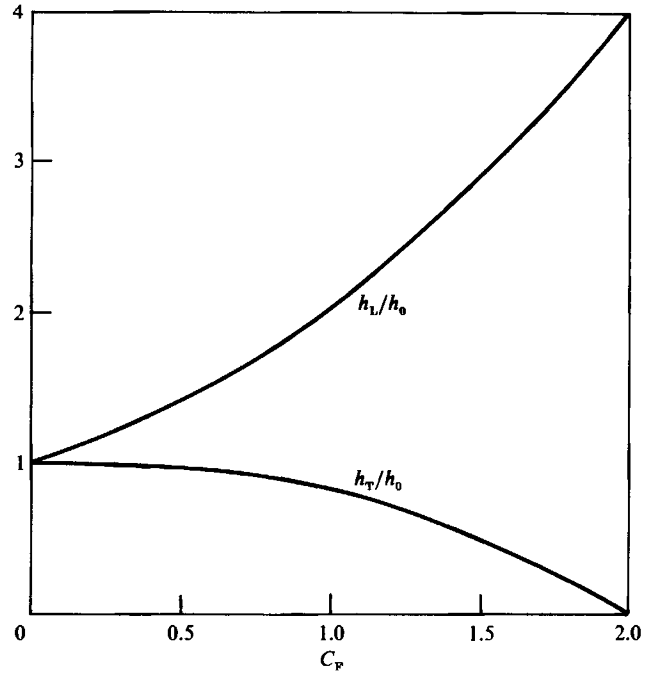

FIGURE 4. Leading- and trailing-edge heights, as functions of the lift coefficient based on distance between centre of pressure and trailing edge.

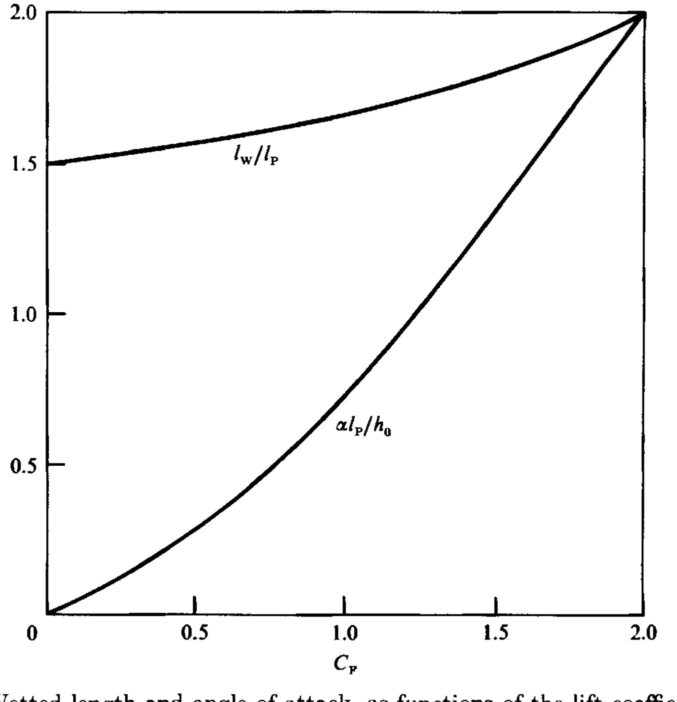

FIGURE 5. Wetted length and angle of attack, as functions of the lift coefficient based on distance between centre of pressure and trailing edge.

water rise, and hence the jet thickness. The trailing edge dips more and more into the undisturbed water, that is, the skimmer skims more.

This continues until the trailing edge hits the bottom, and according to the present theory, this occurs at $C_F = 2$. As we approach that point, the predicted leading-edge height approaches 4 times the undisturbed water height, i.e. the water rises by an amount equal to 3 times its undisturbed height at that point. Since in that limit the flux emerging at the trailing edge tends to zero, all of the incident stream is skimmed into the jet. There is no flow beneath the skimmer; hence the pressure is uniform, and the centre of pressure has moved to mid-chord, i.e. $l_W/l_P = 2.0$. The actual small-velocity flow that remains as this limiting situation is approached is not given in detail by the present asymptotic analysis, but would need a carefully matched theory in which the leading-edge zone is no longer assumed as small as $O(h_0)$.

The skimmer cannot fly if $C_F > 2$. That is, we have found an effective upper bound

$$F < \rho u_0^2 l_P, \tag{5.1}$$

on the weight, or equivalently a lower bound

$$u_0 > \left(\frac{F}{\rho l_P}\right)^{\frac{1}{2}}, \tag{5.2}$$

on the speed at which a skimmer of weight F can fly. Note that l_P is beneficial; you can fly slower by putting your weight further forward.

6. Conclusion

In this paper we have analysed only perhaps the simplest possible model for a surf skimmer, in which viscosity, gravity, finite water depth, and lateral flow, are all neglected. Nevertheless, novel features of this planing problem are treated carefully, including the all-important indeterminancy of the wetted length, and the final results can be used to predict actual dynamics of ridden boards. Extensions of the present work to account for some of the above neglected effects are being considered.

Support by the Australian Research Council is gratefully acknowledged. Discussions with and demonstrations by Max Haselgrove have been particularly valuable to the authors.

Appendix. Exact inner solution

It is convenient to scale the flow of figure 2 by use of the jet thickness h_J as a lengthscale and the incident stream speed u_0 as a velocity scale, defining a non-dimensional complex coordinate

$$z = \pi \frac{x+\mathrm{i}y}{h_J}, \tag{A 1}$$

and a non-dimensional complex velocity potential

$$f(z) = \pi \frac{\phi+\mathrm{i}\psi}{u_0 h_J}. \tag{A 2}$$

Then $f(z)$ is a member of a one-parameter family, the parameter

$$\gamma = \frac{4\lambda}{(1-\lambda)^2} \tag{A 3}$$

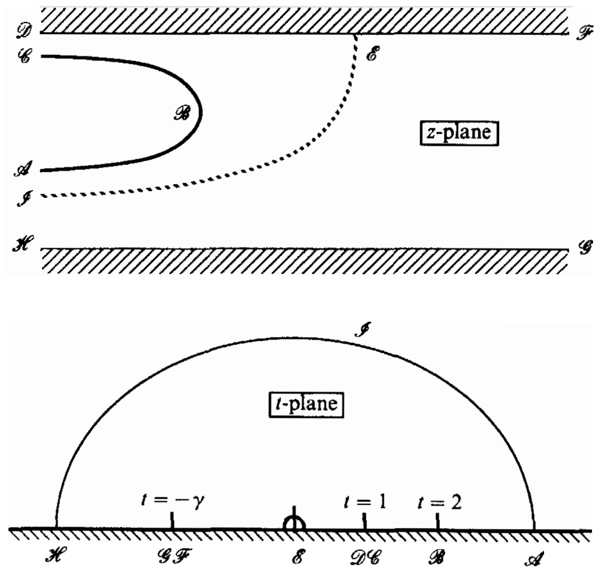

FIGURE 6. Conformal mapping of the leading-edge flow domain.

being related to the contraction coefficient λ. The solution can be obtained by hodograph methods, and the final result is

$$f = -\log(t-1) - \gamma \log(t+\gamma), \tag{A 4}$$

where
$$z = -(\gamma+2)\log(t+\gamma) + \log(t-1) - 4\mathrm{i}(\gamma+1)^{\frac{1}{2}} \arctan\left(\frac{t-1}{\gamma+1}\right)^{\frac{1}{2}}. \tag{A 5}$$

Equation (A 5) maps the region of flow to the upper half t-plane, as shown in figure 6, the correspondence between points $\mathscr{A}$–$\mathscr{I}$ being easy to check directly. The non-dimensional complex velocity is

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\mathrm{d}f}{\mathrm{d}t}\bigg/\frac{\mathrm{d}z}{\mathrm{d}t} = -\frac{t}{(1-\mathrm{i}(t-1)^{\frac{1}{2}})^2}, \tag{A 6}$$

whose magnitude takes a constant (unit) value as required on the free streamline $\mathscr{ABC}$, where t is real and $t > 1$. Also $\mathrm{d}f/\mathrm{d}z$ is real on the remainder of the real t-axis, guaranteeing that these boundaries are horizontal impermeable walls.

The properties of the mapping (A 5) yield immediately relations equivalent to (3.6)–(3.9). For example, the following jumps in the (non-dimensional) y-coordinates are observed:

$$\left.\begin{aligned} [\mathrm{Im}\, z]_{\mathscr{A}}^{\mathscr{D}} &= \pi, \\ [\mathrm{Im}\, z]_{\mathscr{H}}^{\mathscr{I}} &= \pi\gamma, \\ [\mathrm{Im}\, z]_{\mathscr{G}}^{\mathscr{F}} &= \pi(2+\gamma+2(1+\gamma)^{\frac{1}{2}}). \end{aligned}\right\} \tag{A 7}$$

The first of these jumps confirms the scaling (A 1) with respect to the jet thickness h_{J}, the second shows that the parameter γ can be identified with the ratio between the non-jet and jet portions of the incident stream, namely

$$\gamma = h_{\mathrm{T}}/h_{\mathrm{J}}, \tag{A 8}$$

while the last jump is equal to $\pi\gamma/\lambda$, and hence confirms (3.7). Note also that as $t \to -\gamma$, i.e. as we approach the outgoing stream u_L at $\mathscr{F}\mathscr{G}$, (A 6) gives

$$\frac{u_L}{u_0} = \frac{\gamma}{2+\gamma+2(1+\gamma)^{\frac{1}{2}}} = \lambda, \tag{A 9}$$

confirming (3.6).

Since the stagnation point $\mathscr{E}$ occurs at $t = 0$ and the point $\mathscr{B}$ where the free surface is vertical occurs at $t = 2$, we can easily compute the horizontal separation

$$x_{\mathscr{E}} - x_{\mathscr{B}} = \frac{h_J}{\pi}\left[(\gamma+2)\log\frac{\gamma+2}{\gamma} - 2(\gamma+1)^{\frac{1}{2}}\log\frac{(\gamma+1)^{\frac{1}{2}}-1}{(\gamma+1)^{\frac{1}{2}}+1}\right], \tag{A 10}$$

between these points, and as stated in §2, it is necessarily $O(h_J)$ for any finite $\gamma \neq 0$. In the limit as $\gamma \to \infty$ of a very thin jet, the distance given by (A 10) approaches zero with h_J like $6h_J/\pi$. At the other extreme, as $\gamma \to 0$, the stagnation point $\mathscr{E}$ moves far downstream, ultimately merging with $\mathscr{F}$, and the distance given by (A 10) tends to infinity (on the h_J scale). The matched expansions technique formally loses its validity in both limits $\gamma \to 0$ and $\gamma \to \infty$, but nevertheless appears to predict reasonable results at these extremes.

The streamlines shown in figure 2 are exact calculations at $\gamma = 1$, a case in which (A 4) and (A 5) can be solved to yield $z = z(f)$ in closed form.

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